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EXACT PERIODIC AND LOCALIZED SOLUTIONS OF THE EQUATION $h_t = \Delta \ln h$

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New exact regular solutions of the nonlinear-diffusion equation are found. Various types of evolution of certain classes of localized initial perturbations are described. We show that, when a localized distribution in the form of a ring is specified, the instantaneous occurrence of the singularity in its center results from the diffusive spreading.

Introduction. The nonlinear diffusion equation is often used in various applications and has attracted the attention of many investigators. One variant of this equation that describes the evolution of a thermowedge [1], i.e., spreading of thin fluid films over a rigid body surface [2], and some other problems [3–5] is the subject of research of the present paper. We shall consider the equation

$$h_t = \Delta \ln h, \tag{1}$$

where the subscript denotes differentiation with respect to time and Δ is the three-dimensional Laplacian. According to the accepted terminology, Eq. (1) is the limiting form of the fast-diffusion equation. As far as we know, the exact solutions of Eq. (1) described in the literature have singularities, which make their physical interpretation difficult. We shall consider periodic and localized positive solutions, since they are of the greatest interest from the viewpoint of applications. From the variety of possible exact solutions, we confine our attention to those which can be written explicitly and demonstrate most clearly the special properties of the equation considered.

One-Dimensional Solutions. (1) In considering nonstationary solutions which depend on only one coordinate, it is natural to study self-similar regimes of the evolution of initial perturbations. Equation (1) is known [2, 5] to admit a class of self-similar solutions with an arbitrary self-similar index that controls the scale of coordinate extension with time. If we restrict ourselves to the case where self-similar equations take the form of a conservation law, one of the possible solutions can be written in the form

$$h = (\tau + t)^{-1} H(\xi), \qquad \xi = x(\tau + t)^{-1} \tag{2}$$

(τ is an arbitrary constant). Substituting (2) into (1) and performing identical manipulations, we obtain

$$\frac{\partial}{\partial \xi}\Big[H\Big(\frac{\partial}{\partial \xi}\Big(\frac{1}{H}\Big) - \xi + \frac{\alpha}{H}\Big)\Big] = 0,$$

where α is a constant. Integrating, we obtain a linear equation for the function inverse to H, which enables us to find a localized solution of the form

$$h = \frac{\alpha^2}{(\tau + t)}\Big[C \exp\Big(-\frac{\alpha x}{\tau + t}\Big) - 1 + \frac{\alpha x}{\tau + t}\Big]^{-1}. \tag{3}$$

In order that this solution have no singularities, the constant C must be greater than unity. If α is equal to zero, Eq. (3) takes the form

$$h = \frac{2(\tau + t)}{2C(\tau + t)^2 + x^2}.$$

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The above solution describes the spreading of a one-dimensional layer and is regular at any moment of time. In the asymmetrical case, the maximum of perturbation moves with constant speed towards the more shallow front. It should be noted that here and henceforth the time is reckoned from the zero value. The arbitrary constants which enter (4) determine the characteristic dimension and the amplitude of the initial perturbation.

(2) Another solution describes the uniformly moving front and has the form [1]

$$h = H(\xi), \qquad \xi = x - \beta t. \tag{4}$$

Substituting (4) into Eq. (1) and integrating, we obtain

$$\frac{\partial}{\partial \xi}\left(\frac{1}{H}\right) = \beta - \frac{\alpha}{H},$$

where α and β are arbitrary constants. Writing the general solution and satisfying the regularity conditions, we have

$$h = \frac{\alpha^2}{\alpha\beta + \exp\left(-\alpha(x - \beta t)\right)}. \tag{5}$$

It is noteworthy that to satisfy the condition of the absence of singularities, the constants α and β should be of the same sign. It follows from the analysis of (5) that the front moves in the direction of the decreasing thickness of the fluid layer, and the constants determine the width of the front, its velocity, and the thickness of the perturbed fluid layer.

(3) To describe self-similar regimes which correspond to the spreading of a line or cylinder, we introduce the cylindrical coordinate system and use the transformation

$$h(r, \varphi, t) = r^{-2} h(x, y, t), \qquad x = \ln r, \quad y = \varphi. \tag{6}$$

We note that, for the axisymmetrical case, this transformation was used in [2]. Using (6), we obtain the following formulas for the case of cylindrical symmetry:

$$h = \frac{\alpha^2 r^{\alpha-2}}{r^\alpha + C\exp(t)}; \tag{7}$$

$$h = \frac{\alpha m r^{m-2}}{C + r^m(m\ln r - 1)}, \qquad m = \frac{\alpha}{\tau + t}. \tag{8}$$

The first solution describes the slow spreading of a localized spot or ring and does not differ qualitatively from the above diffusive regimes of evolution of a plane layer. The lifetime of this perturbation is unbounded and does not depend on the characteristic dimensions of the initial perturbation. For $\alpha < 2$, expression (7) describes the evolution of the line which has a singularity at the coordinate origin. Solution (8) is of significant interest, since it demonstrates the possibility of instantaneous rearrangement of the perturbation structure at the moment which corresponds to $m = 2$. In contrast to the first regime, the ring shrinks up to a certain critical moment of time, which leads to the cumulation and instantaneous alteration of the structure for $m = 2$. At this moment, a singularity occurs in the center of the ring, which then damps out hyperbolically. In this case, time plays the role of a bifurcational parameter. This character of solution corresponds to jet ejection of the fluid when a heavy body falls on it. This cumulative effect is caused by the above mentioned property of asymmetrical perturbations (4) to displace in the direction of the slower decrease in the layer thickness.

(4) In the three-dimensional case where the point is the evolution of the spherical perturbation, one can construct a self-similar solution of the form $h = (\tau - t)^3 H(\xi)$ and $\xi = r(\tau - t)$. Integrating, we obtain the equation for the function H:

$$\xi^2 \frac{\partial}{\partial \xi}\left(\frac{1}{H}\right) = -\frac{\alpha}{H} + \xi^3. \tag{9}$$

The solution of this equation reduces to quadratures, but it cannot be expressed via elementary and special functions. Solving this equation numerically, we infer that if the integration constant α is positive, the solution has one maximum and the following asymptotes:

$$H = 2/\xi^2, \quad \xi \to \infty; \qquad H = \exp(-\alpha/\xi), \quad \xi \to 0.$$

For zero values of α, there exists an analytical solution, the particular form of which was found in [2]:

$$h = \frac{2(\tau - t)^3}{2C + (\tau - t)^2(x^2 + y^2 + z^2)}.$$

Multidimensional Solutions. (1) We seek asymmetrical solutions in the form

$$h = (\tau - t)H(x, y, z). \tag{10}$$

Substituting (10) into (1), we obtain the stationary equation

$$\Delta \ln H = -H. \tag{11}$$

In the two-dimensional case, Eq. (11) coincides with the Liouville equation, whose general solution is expressed in terms of an arbitrary complex function [6]. The types of possible solutions of Eq. (11) were discussed in many papers, for example, in [6–8] (see also the references cited therein); therefore, we do not dwell upon the details of analysis here and write only the two-dimensional periodic localized solutions that we are aware of:

$$H = 2N^2(1 - \varepsilon^2)/(\cosh(Nx) + \varepsilon \sin(Ny))^2; \tag{12}$$

$$H = 8N^2(1 - \varepsilon^2)r^{2N-2}/(1 + r^{2N} + 2\varepsilon r^N \sin(N\varphi))^2. \tag{13}$$

Here ε is a constant having the meaning of a bifurcation parameter.

The first solution is known in the shallow water theory, and the second solution was found probably in describing stationary structures in the plasma [7]. We note that these solutions change to one another by means of the transformation (6). If $\varepsilon = 0$, formulas (12) and (13) describe one more type of evolution of plane and axisymmetrical initial perturbations. The fact that the lifetime of the periodic structures described by solutions (12) and (13) can exceed the characteristic diffusion period is worth noting. This enables us to interpret them as quasistationary formations which can collapse during a period which is much lesser than the characteristic diffusion period for certain relations between the amplitude and characteristic dimension of the initial perturbation.

(2) We shall seek three-dimensional nonsymmetrical solutions of Eq. (1) in the form

$$h = (\tau - t)r^{-2}H(\theta, \varphi),$$

where r, θ, and φ are the spherical coordinates. We note that in contrast to the above solutions, this solution has a singularity at the coordinate origin and describes the decay of the delta function. Substituting the indicated expression into (1), we obtain

$$-H = -2 + \sin^{-2}\theta\left[\sin\theta \frac{\partial}{\partial\theta}\sin\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\varphi^2}\right]\ln H. \tag{14}$$

We find the solution of Eq. (14) in the form

$$H = \sin^{-2}\theta F(x, y), \qquad x = \ln(\tan(\theta/2)), \qquad y = \varphi.$$

As a result, Eq. (14) reduces to the Liouville equation [6] written in the Cartesian coordinates, which makes it possible to use formula (12) as one of the solutions. Passing to the spherical coordinates, we finally have

$$h = \frac{\tau - t}{r^2}\,\frac{8N^2(1 - \varepsilon^2)\sin^{2N-2}\theta}{[(1 + \cos\theta)^N + (1 - \cos\theta)^N + 2\varepsilon \sin^N\theta \sin(N\varphi)]^2}, \tag{15}$$

where N is an integer and ε is an arbitrary constant the maximum value of which is equal to unity. Solution (15) describes toroid-like structures. For $N = 1$ and $\varepsilon = 0$, it coincides with the solution of [3], which was obtained in studying the evolution of the singular axisymmetrical initial distribution.

Conclusions. We note that, probably, the only stationary regular solution of Eq. (1) exists only in the three-dimensional case and can be presented as follows:

$$h = C\exp(-1/r), \qquad r^2 = x^2 + y^2 + z^2.$$

It is of interest to find a criterion which can be used to determine the character of damping. According to the results, the localized perturbation can spread slowly or disappear for a certain characteristic time determined by the amplitude and dimensions of the perturbation. The possibility of instantaneous occurrence of a singularity at the center of the spreading ring has been established.

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